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## A NUMERICAL CRITERION FOR ADMISSIBILITY OF SEMI-SIMPLE ELEMENTS

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**Abstract.** In this article, we shall generalize a theorem of Cattani and Kaplan on horizontal representations of  $SL(2)$ . Their theorem plays an important role in the construction of their partial compactifications of the classifying spaces  $D$  modulo an arithmetic subgroup of Hodge structures of weight 2.

**Introduction.** A horizontal  $SL_2$ -representation is a generalization of the notion of “ $(H_1)$ -homomorphism” of  $SL_2$  in the case of the classical theory of Hermitian symmetric domains (cf., e.g., [Sa, III]). More precisely, let  $G = G_{\mathbf{R}} := \text{Aut}(H_{\mathbf{R}}, S)$  be the automorphism group of the classifying space  $D$  of Hodge structures of weight  $w$  (see §1). A representation  $\rho: SL_2(\mathbf{R}) \rightarrow G$  is said to be *horizontal* at  $r \in D$  if the morphism  $\rho_*: \mathfrak{sl}_2(\mathbf{R}) \rightarrow \mathfrak{g}$  of the Lie algebras is a morphism of Hodge structures of type  $(0, 0)$  with respect to the Hodge structures on  $\mathfrak{sl}_2(\mathbf{C})$  and  $\mathfrak{g}_{\mathbf{C}}$  induced by  $i \in U := (\text{upper-half plane})$  and  $r \in D$  respectively (see Definition (2.1)). In this case, the pair  $(\rho, r)$  is uniquely determined by the pair  $(Y, r) \in \mathfrak{g} \times D$  with

$$(0.1) \quad Y := \rho_* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Conversely, a pair  $(Y, r) \in \mathfrak{g} \times D$  is said to be *admissible* if there exists a representation  $\rho: SL_2(\mathbf{R}) \rightarrow G$  horizontal at  $r$  and satisfying (0.1). The main result in the present article is a numerical criterion for admissibility of a pair  $(Y, r)$  in the case of general weight.

Given a pair  $(\rho, r)$  as above, one can refine the Hodge decomposition  $H_{\mathbf{C}} = \bigoplus H_r^{p,q}$ , corresponding to  $r \in D$ , under the horizontal action of  $\mathfrak{sl}_2(\mathbf{C})$  at  $r$ , called a *Hodge- $(Z, X_{\pm})$  decomposition* (see (2.7)). Our proof of the main result is based on an elementary but useful observation (Corollary (2.11), see also Remark (2.12)), which says that the transformation of the Hodge- $(Z, X_{\pm})$  decomposition by the inverse  $c^{-1}$  of the Cayley element

$$c := \rho \left( \exp \frac{\pi i}{4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$$

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yields a split mixed Hodge structure, called a *mixed Hodge- $(Y, N_{\pm})$  decomposition*, which is nothing but the limiting mixed Hodge structure of the associated  $SL_2$ -orbit  $\tilde{\rho}: U \rightarrow D$  defined by  $\tilde{\rho}(gi) := \rho(g)r$  for  $g \in SL_2(\mathbf{R})$  (cf. [Sc, Theorem (6.16)] and its proof). By virtue of this observation, we can view the relationship between the pairs  $(\rho, r)$  and  $(Y, r)$  from a better perspective, and generalize a numerical criterion [CK, Theorem (2.22)] for admissibility of  $(Y, r)$  in the case of weight 2 to the case of general weight.

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**1. Preliminaries.** We recall first the definition of a (polarized) Hodge structure of weight  $w$ . Fix a free  $\mathbf{Z}$ -module  $H_{\mathbf{Z}}$  of finite rank. Set  $H_{\mathbf{Q}} := \mathbf{Q} \otimes H_{\mathbf{Z}}$ ,  $H = H_{\mathbf{R}} := \mathbf{R} \otimes H_{\mathbf{Z}}$  and  $H_{\mathbf{C}} := \mathbf{C} \otimes H_{\mathbf{Z}}$ , whose complex conjugation is denoted by  $\sigma$ . Let  $w$  be an integer. A *Hodge structure of weight  $w$*  on  $H_{\mathbf{C}}$  is a decomposition

$$(1.1) \quad H_{\mathbf{C}} = \bigoplus_{p+q=w} H^{p,q} \quad \text{with} \quad \sigma H^{p,q} = H^{q,p}.$$

The integers

$$(1.2) \quad h^{p,q} := \dim H^{p,q}$$

are called the Hodge numbers.

A polarization  $S$  for a Hodge structure (1.1) of weight  $w$  is a non-degenerate bilinear form on  $H_{\mathbf{Q}}$ , symmetric if  $w$  is even and skew-symmetric if  $w$  is odd, such that its  $\mathbf{C}$ -bilinear extension, denoted also by  $S$ , satisfies

$$(1.3) \quad \begin{aligned} S(H^{p,q}, \sigma H^{p',q'}) &= 0 \quad \text{unless} \quad (p, q) = (p', q'), \\ i^{p-q} S(v, \sigma v) &> 0 \quad \text{for all} \quad 0 \neq v \in H^{p,q}. \end{aligned}$$

**REMARK (1.4).** In the geometric case, i.e., the Hodge structure on the  $w$ -th cohomology group  $H^w(X, \mathbf{Q})$  of a smooth projective variety  $X \subset \mathbf{P}^N$  of dimension  $d$  over  $\mathbf{C}$ , we take as a polarization

$$S(u, v) := (-1)^{w(w-1)/2} \int_X u \wedge v \wedge \eta^{d-w}$$

for primitive classes  $u, v \in H_{\text{prim}}^w(X, \mathbf{C}) \simeq H_{\text{prim}}^w(X, \Omega_X^1)$  where  $\eta \in H^1(X, \Omega_X^1)$  is the cohomology class of a hyperplane section of  $X$ .

For fixed  $S$  and  $\{h^{p,q}\}$ , the classifying space  $D$  for Hodge structures and its “compact dual”  $\check{D}$  are defined by

$$\begin{aligned}
 \check{D} &:= \{ \{H^{p,q}\} \mid \text{Hodge structure on } H_{\mathbf{C}} \text{ with } \dim H^{p,q} = h^{p,q}, \\
 &\quad \text{satisfying the first condition in (1.3)} \}, \\
 (1.5) \quad D &:= \{ \{H^{p,q}\} \in \check{D} \mid \text{satisfying also the second condition in (1.3)} \}.
 \end{aligned}$$

These are homogeneous spaces under the natural actions of the groups

$$(1.6) \quad G_{\mathbf{C}} := \text{Aut}(H_{\mathbf{C}}, S), \quad G = G_{\mathbf{R}} := \{g \in G_{\mathbf{C}} \mid gH_{\mathbf{R}} = H_{\mathbf{R}}\},$$

respectively. Taking a reference point  $r \in D$ , one obtains identifications

$$(1.7) \quad \check{D} \simeq G_{\mathbf{C}}/B_{\mathbf{C}}, \quad D \simeq G/V,$$

where  $B_{\mathbf{C}}$  and  $V$  are the isotropy subgroups of  $G_{\mathbf{C}}$  and of  $G$  at  $r \in D$ , respectively. It is a direct consequence of the definition that

$$(1.8) \quad G \simeq \begin{cases} O(2h, k), & V \simeq \begin{cases} U(h^{w,0}) \times \cdots \times U(h^{t+1,t-1}) \times O(h^{t,t}) & \text{if } w=2t, \\ U(h^{w,0}) \times \cdots \times U(h^{t+1,t}) & \text{if } w=2t+1, \end{cases} \end{cases}$$

where  $k := \sum_{|j| \leq [t/2]} h^{t+2j,t-2j}$  and  $h := (\dim H - k)/2$  if  $w=2t$ , and  $h := \dim H/2$  if  $w=2t+1$ . It is an important observation that  $V$  is compact, but not maximal compact in general. Hence  $D$  is a symmetric domain of Hermitian type if and only if

$$(1.9) \quad h^{p,q} = 0 \quad \text{unless } (p, q) = \begin{cases} (t+1, t-1), & (t, t) \quad \text{or} \quad (t-1, t+1), \\ & \text{and } h^{t+1,t-1} = 1 \quad \text{if } w=2t, \\ (t+1, t) \quad \text{or} \quad (t, t+1) & \text{if } w=2t+1. \end{cases}$$

A reference Hodge structure  $r = \{H_r^{p,q}\} \in D$  induces a Hodge structure of weight 0 on the Lie algebra  $\mathfrak{g}_{\mathbf{C}} := \text{Lie } G_{\mathbf{C}}$  by

$$(1.10) \quad \mathfrak{g}_{\mathbf{C}}^{s,-s} := \{X \in \mathfrak{g}_{\mathbf{C}} \mid XH_r^{p,q} \subset H_r^{p+s,q-s} \text{ for all } p, q\}.$$

One can define the associated Cartan involution  $\theta_r$  on  $\mathfrak{g}_{\mathbf{C}}$  by

$$(1.11) \quad \theta_r(X) := \sum_s (-1)^s X^{s,-s} \quad \text{for } X = \sum_s X^{s,-s} \in \mathfrak{g}_{\mathbf{C}} = \bigoplus_s \mathfrak{g}_{\mathbf{C}}^{s,-s}.$$

This can be interpreted in the following way: Set

$$\begin{aligned}
 (1.12) \quad H_r^+ &:= H_r^{w,0} \oplus H_r^{w-2,2} \oplus H_r^{w-4,4} \oplus \cdots, \\
 H_r^- &:= H_r^{w-1,1} \oplus H_r^{w-3,3} \oplus H_r^{w-5,5} \oplus \cdots.
 \end{aligned}$$

It is clear by definition that the isotropy subgroup of the decomposition  $H_{\mathbf{C}} = H_r^+ \oplus H_r^-$  induces the maximal compact subgroup

$$(1.13) \quad K \simeq \begin{cases} O(2h) \times O(k) & \text{if } w=2t, \\ U(h) & \text{if } w=2t+1, \end{cases}$$

of  $G$  which contains  $V$ , and the Cartan involution  $\theta_r$  in (1.11) is the one associated to

K. Define a  $C$ -linear automorphism

$$(1.14) \quad E_r: H_C \rightarrow H_C \quad \text{by} \quad E_r := \begin{cases} 1 & \text{on } H_r^+, \\ -1 & \text{on } H_r^-. \end{cases}$$

Then the Cartan involution  $\theta_r$  in (1.11) can also be written as

$$(1.15) \quad \theta_r X = (\text{Ad } E_r) X \quad \text{for } X \in \mathfrak{g}_C.$$

We recall now well-known results on  $SL_2$ -representations. Let  $\xi, \eta$  be two variables, and write

$$(1.16) \quad \begin{pmatrix} \xi \\ \eta \end{pmatrix}^{(m)} := \begin{pmatrix} \xi^m \\ \xi^{m-1}\eta \\ \vdots \\ \eta^m \end{pmatrix} \quad (m=0, 1, 2, \dots).$$

A representation

$$(1.17) \quad \rho_m: SL_2(\mathbf{R}) \rightarrow SL_{m+1}(\mathbf{R}) \quad \text{defined by} \quad \rho_m(g) \begin{pmatrix} \xi \\ \eta \end{pmatrix}^{(m)} := \left( g \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right)^{(m)}$$

is called a symmetric tensor representation of dimension  $m+1$ . It is known that the  $\rho_m$  ( $m=0, 1, 2, \dots$ ) are absolutely irreducible and constitute a full set of representatives for the equivalence classes of finite dimensional irreducible representations of  $SL_2(\mathbf{R})$ .

We take the standard generators for the Lie algebras  $\mathfrak{sl}_2(\mathbf{R})$  and  $\mathfrak{su}(1, 1)$  which are related by the Cayley transformation  $\text{Ad } c_1$ , where

$$(1.18) \quad c_1 := \exp \frac{\pi i}{4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix},$$

as follows:

$$(1.19) \quad \begin{array}{ccccc} \mathfrak{sl}_2(\mathbf{R}) & \ni y := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & n_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & n_- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \text{Ad } c_1 \downarrow & \downarrow & \downarrow & \downarrow \\ \mathfrak{su}(1, 1) & \ni z := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, & x_+ := \frac{1}{2} \begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}, & x_- := \frac{1}{2} \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}. \end{array}$$

The following lemma can be verified directly by using the monomial basis (1.16) and the definition (1.19), and so we omit the proof.

LEMMA (1.20). (i) In the above notation,  $Y_m := \rho_{m*}(y)$  and  $N_{m\pm} := \rho_{m*}(n_{\pm})$  satisfy

$$\begin{aligned} Y_m(\xi^{m-j}\eta^j) &= (m-2j)\xi^{m-j}\eta^j, \\ N_{m+}(\xi^{m-j}\eta^j) &= (m-j)\xi^{m-j-1}\eta^{j+1}, \\ N_{m-}(\xi^{m-j}\eta^j) &= j\xi^{m-j+1}\eta^{j-1}. \end{aligned}$$

- (ii) For the Cayley element  $c_m := \rho_m(c_1) \in SL_{m+1}(C)$ ,  
 $\sigma c_m = c_m^{-1} \sigma$ , where  $\sigma$  is the complex conjugation.  
 $c_m^{\pm 2}(\xi^{m-j} \eta^j) = (\pm i)^m \eta^{m-j} \xi^j$ ,  
 $c_m^4(\xi^{m-j} \eta^j) = (-1)^m \xi^{m-j} \eta^j$ .

REMARK (1.21). The Hodge structure on  $\mathfrak{g}_{1C} := \mathfrak{sl}_2(C)$  induced by  $i \in U := (\text{upper-half plane}) \simeq SL_2(\mathbf{R})/U(1)$  coincides with the canonical decomposition by the standard “H-element”  $(n_+ - n_-)/2$  (cf., e.g., [Sa, II, §7]):

$$\mathfrak{g}_{1C} = \mathfrak{g}_{1C}^{1,-1} + \mathfrak{g}_{1C}^{0,0} + \mathfrak{g}_{1C}^{-1,1} = \mathfrak{p}_- + \mathfrak{k}_C + \mathfrak{p}_+ = \{x_-\}_C + \{z\}_C + \{x_+\}_C.$$

**2. Horizontal  $SL_2$ -representations.** From now on, we assume that  $w > 0$  and all Hodge structures of weight  $w$  satisfy  $H^{p,q} = 0$  unless  $p, q \geq 0$ .

DEFINITION (2.1) (cf. [Sc, p. 258]). An  $SL_2$ -representation  $\rho: SL_2(\mathbf{R}) \rightarrow G$  is said to be horizontal at  $r = \{H_r^{p,q}\} \in D$  if  $\rho_*(x_+) \in \mathfrak{g}_C^{-1,1} := \{X \in \mathfrak{g}_C \mid XH_r^{p,q} \subset H_r^{p-1,q+1} \text{ for all } p, q\}$ .

REMARK (2.2). It is clear that an  $SL_2$ -representation  $\rho$  is horizontal if and only if  $\rho_*: \mathfrak{sl}_2(\mathbf{R}) \rightarrow \mathfrak{g}$  is a morphism of Hodge structures of type  $(0, 0)$  with respect to the Hodge structures induced by  $i \in U$  and  $r \in D$ , respectively. A horizontal  $SL_2$ -representation  $\rho$  induces an equivariant horizontal map  $\tilde{\rho}: P^1 \rightarrow \check{D}$  with  $\tilde{\rho}(i) = r$ :

$$\begin{array}{ccc} SL_2(C) & \xrightarrow{\rho} & G_C \\ \downarrow & & \downarrow \\ P^1 & \xrightarrow{\tilde{\rho}} & \check{D}. \end{array}$$

This is a generalization to the present context of the notion of ‘ $(H_1)$ -homomorphism’ in the case of symmetric domains of Hermitian type (cf., e.g., [Sa, II, (8.5), III, §1]).

Let  $\rho: SL_2(\mathbf{R}) \rightarrow G$  be a representation horizontal at  $r \in \{H_r^{p,q}\} \in D$ , and set

$$(2.3) \quad Y := \rho_*(y), \quad N_{\pm} := \rho_*(n_{\pm}); \quad Z := \rho_*(z), \quad X_{\pm} := \rho_*(x_{\pm}).$$

Notice that by (1.19) these are related under the Cayley transformation:

$$(2.4) \quad Z = (\text{Ad } c)Y, \quad X_{\pm} = (\text{Ad } c)N_{\pm}, \quad c := \rho(c_1).$$

$(Y, N_{\pm})$  and  $(Z, X_{\pm})$  define direct sum decompositions of  $H$  and  $H_C$  whose summands are

$$(2.5) \quad P_{\lambda}^{(\lambda+2k)} := N_{-}^k(H(Y; \lambda+2k) \cap \text{Ker } N_{+}),$$

$$(2.6) \quad Q_{\lambda}^{(\lambda+2k)} := X_{-}^k(H_C(Z; \lambda+2k) \cap \text{Ker } X_{+}),$$

for all eigenvalues  $\lambda \in \{0, \pm 1, \pm 2, \dots, \pm w\}$  of  $Y$  and  $Z$  and for  $k \geq \max\{-\lambda, 0\}$ , respectively. Here we denote by  $H(Y; \lambda+2k)$  etc. the eigenspace of an endomorphism

$Y$  of  $H$  with eigenvalue  $\lambda + 2k$ . Since  $\rho$  is horizontal at  $r = \{H_r^{p,q}\}$ , (2.6) is compatible with this Hodge structure and we set

$$(2.7) \quad Q_\lambda^{(\lambda+2k)a+k, b+\lambda+k} := Q_\lambda^{(\lambda+2k)} \cap H_r^{a+k, b+\lambda+k} \quad (a, b \geq 0).$$

These form a refined direct sum decomposition which we call the *Hodge- $(Z, X_\pm)$  decomposition* of  $(\rho, r)$  (cf. Remark (2.12) below). Transforming this by the inverse  $c^{-1}$  of the Cayley element, we define

$$(2.8) \quad P_\lambda^{(\lambda+2k)a+k, b+\lambda+k} := c^{-1} Q_\lambda^{(\lambda+2k)a+k, b+\lambda+k}.$$

LEMMA (2.9). (i)  $\sigma Q_\lambda^{(\lambda+2k)a+k, b+\lambda+k} = Q_{-\lambda}^{(-\lambda+2(\lambda+k))b+\lambda+k, a+k}$ .  
(ii)  $c Q_\lambda^{(\lambda+2k)a+k, b+\lambda+k} = c^2 P_\lambda^{(\lambda+2k)a+k, b+k} = P_{-\lambda}^{(-\lambda+2(\lambda+k))a+\lambda+k, b+\lambda+k}$ .  
 $c^{-1} P_\lambda^{(\lambda+2k)a+k, b+k} = c^{-2} Q_\lambda^{(\lambda+2k)a+k, b+\lambda+k} = Q_{-\lambda}^{(-\lambda+2(\lambda+k))a+\lambda+k, b+k}$ .

PROOF. It is easy to see, by definition, that  $c P_\lambda^{(\lambda+2k)} = Q_\lambda^{(\lambda+2k)}$ . Hence, by the first equality in (1.20.ii), we have

$$\sigma Q_\lambda^{(\lambda+2k)} = \sigma c P_\lambda^{(\lambda+2k)} = c^{-1} \sigma P_\lambda^{(\lambda+2k)} = c^{-1} P_\lambda^{(\lambda+2k)} = c^{-2} Q_\lambda^{(\lambda+2k)}.$$

On the other hand, by the second equality in (1.20.ii), the third and the second equalities in (1.20.i), we see that on  $P_\lambda^{(\lambda+2k)}$

$$c^{-2} = \begin{cases} (-i)^{\lambda+2k} \frac{k!}{(\lambda+k)!} N_-^\lambda & \text{if } \lambda \geq 0, \\ (-i)^{\lambda+2k} \frac{(\lambda+k)!}{k!} N_+^{-\lambda} & \text{if } \lambda < 0. \end{cases}$$

Taking their Cayley transforms, we see that on  $Q_\lambda^{(\lambda+2k)}$

$$(2.10) \quad c^{-2} = \begin{cases} (-i)^{\lambda+2k} \frac{k!}{(\lambda+k)!} X_-^\lambda & \text{if } \lambda \geq 0, \\ (-i)^{\lambda+2k} \frac{(\lambda+k)!}{k!} X_+^{-\lambda} & \text{if } \lambda < 0. \end{cases}$$

Thus, by the definition of the  $Q_\lambda^{(\lambda+2k)}$ , we have in both cases that

$$\sigma Q_\lambda^{(\lambda+2k)} = c^{-2} Q_\lambda^{(\lambda+2k)} = X_{\mp}^{\pm \lambda} Q_\lambda^{(\lambda+2k)} = Q_{-\lambda}^{(\lambda+2k)} = Q_{-\lambda}^{(-\lambda+2(\lambda+k))}.$$

This together with  $\sigma H_r^{a+k, b+\lambda+k} = H_r^{b+\lambda+k, a+k}$  yields the assertion (i).

By horizontality,  $X_- \in \mathfrak{g}_c^{1,-1}$  and  $X_+ \in \mathfrak{g}_c^{-1,1}$ , hence  $X_{\mp}^{\pm \lambda} \in \mathfrak{g}_c^{\lambda, -\lambda}$ . This together with (2.10) shows that

$$c^{-2} Q_\lambda^{(\lambda+2k)a+k, b+\lambda+k} = X_{\mp}^{\pm \lambda} Q_\lambda^{(\lambda+2k)a+k, b+\lambda+k} = Q_{-\lambda}^{(-\lambda+2(\lambda+k))a+\lambda+k, b+k}.$$

Thus we obtain the second equality in (ii). The first equality in (ii) follows from the second. ■

COROLLARY (2.11). Let  $(\rho, r)$  be as above. For each eigenvalue  $\lambda$  of  $Y$  and for  $k \geq \max\{-\lambda, 0\}$ , we see that

$$C \otimes P_{\lambda}^{(\lambda+2k)} = \bigoplus_{\substack{a+b+2k=w-\lambda \\ a, b \geq 0}} P_{\lambda}^{(\lambda+2k)a+k, b+k}$$

is a Hodge structure of weight  $w-\lambda$ . Moreover, in the case  $\lambda=k=0$ , this is  $S$ -polarized.

PROOF. We should observe the behavior under the complex conjugation  $\sigma$ :

$$\begin{aligned} \sigma P_{\lambda}^{(\lambda+2k)a+k, b+k} &= \sigma c^{-1} Q_{\lambda}^{(\lambda+2k)a+k, b+\lambda+k} = c Q_{-\lambda}^{(-\lambda+2(\lambda+k))b+\lambda+k, a+k} \\ &= c^2 P_{-\lambda}^{(-\lambda+2(\lambda+k))b+\lambda+k, a+\lambda+k} = P_{\lambda}^{(\lambda+2k)b+k, a+k}. \end{aligned}$$

This shows the first assertion.

The representation  $\rho$  is trivial on  $Q_0^{(0)}$ , hence  $P_0^{(0)a, b} = c^{-1} Q_0^{(0)a, b} = Q_0^{(0)a, b}$ , and so the second assertion trivially holds. ■

We call a direct sum decomposition in (2.11) the *mixed Hodge- $(Y, N_{\pm})$  decomposition* of  $(\rho, r)$ .

REMARK (2.12). We remark here some observations which are verified easily by (1.20.i), their Cayley transforms and horizontality of  $\rho$  at  $r$ . A Hodge- $(Z, X_{\pm})$  decomposition and a mixed Hodge- $(Y, N_{\pm})$  decomposition form “nests of diamonds”, respectively. For example, in the case of weight  $w=3$ , these nests of diamonds are illustrated respectively as in Figures 1 and 2.

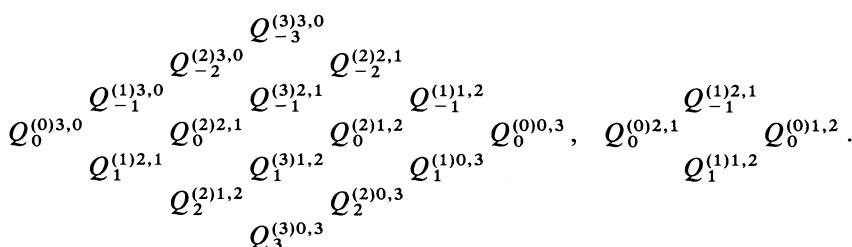


FIGURE 1.

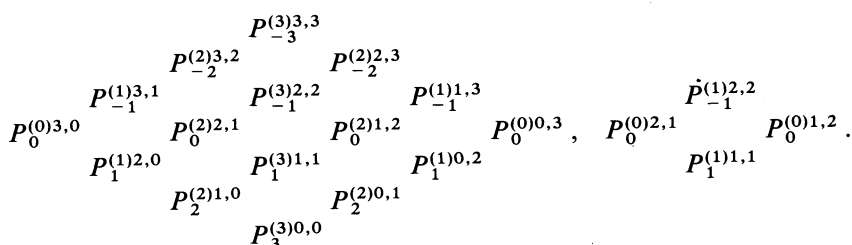


FIGURE 2.



On these nests of diamonds, the complex conjugation by  $\sigma$  sends respectively a summand  $Q_{\lambda}^{(\lambda+2k)a+k, b+\lambda+k}$  to a summand  $Q_{-\lambda}^{(-\lambda+2(\lambda+k))b+\lambda+k, a+k}$  which are symmetric with respect to the origin of the diamonds, and a summand  $P_{\lambda}^{(\lambda+2k)a+k, b+k}$  to a summand  $P_{\lambda}^{(\lambda+2k)b+k, a+k}$  which are symmetric with respect to the vertical axis. The operator  $X_+$  (resp.  $X_-$ ) sends a summand  $Q_{\lambda}^{(\lambda+2k)a+k, b+\lambda+k}$  one step down (resp. up) to a summand  $Q_{\lambda+2}^{(\lambda+2+2(k-1))a+k-1, b+\lambda+2+k-1}$  (resp.  $Q_{\lambda-2}^{(\lambda-2+2(k+1))a+k+1, b+\lambda-2+k+1}$ ), and  $X_{\pm}$  are inverse to each other up to non-zero constant between these summands whenever both summands actually appear in the nest of diamonds. Similarly, the operator  $N_+$  (resp.  $N_-$ ) sends a summand  $P_{\lambda}^{(\lambda+2k)a+k, b+k}$  one step down (resp. up) to a summand  $P_{\lambda+2}^{(\lambda+2+2(k-1))a+k-1, b+k-1}$  (resp.  $P_{\lambda-2}^{(\lambda-2+2(k+1))a+k+1, b+k+1}$ ), and  $N_{\pm}$  are inverse to each other up to non-zero constant between these summands whenever both summands actually appear in the nest of diamonds. The Cayley element  $c$  transforms the second nest of diamonds together with the action of the operators  $Y, N_{\pm}$  to the first nest of diamonds together with the action of the operators  $Z, X_{\pm}$ :  $cP_{\lambda}^{(\lambda+2k)a+k, b+k} = Q_{\lambda}^{(\lambda+2k)a+k, b+\lambda+k}$ .

By using these operators, we can explain why the summands outside the nests of diamonds vanish in the following way. We claim first that  $Q_{\lambda}^{(\lambda+2k)a+k, b+\lambda+k} = 0$  for  $\lambda > 0$  and  $b < 0$ . Indeed,  $X_-^{\lambda+k}$  is injective on this summand by the Cayley transform of the third equality in (1.20.i). On the other hand, looking at the Hodge type, we see that  $X_-^{\lambda+k} Q_{\lambda}^{(\lambda+2k)a+k, b+\lambda+k} \subset Q_{-\lambda-2k}^{(-\lambda-2k+2(\lambda+2k))a+\lambda+2k, b} = 0$  by horizontality. Thus we get our claim. It follows by symmetry under the complex conjugation  $\sigma$  that  $Q_{\lambda}^{(\lambda+2k)a+k, b+\lambda+k} = 0$  for  $\lambda < 0$  and  $a < 0$ . Finally, by the inverse of the Cayley transformation, we have  $P_{\lambda}^{(\lambda+2k)a+k, b+k} = 0$  for  $\lambda > 0$  and  $b < 0$ , and for  $\lambda < 0$  and  $a < 0$ .

We call the length of the side of the biggest diamond in a nest the *size* of the nest of diamonds.

Another remark is that a mixed Hodge- $(Y, N_{\pm})$  decomposition is nothing but the limiting split mixed Hodge structure of the associated  $SL_2$ -orbit  $\tilde{\rho}: U \rightarrow D$ ,  $\tilde{\rho}(gi) := \rho(g)r$  ( $g \in SL_2(\mathbf{R})$ ), and the monodromy weight filtration  $L$  is described as  $L_i = \bigoplus_{\lambda \leq i} \bigoplus_k P_{-\lambda}^{(\lambda+2k)}$  (cf. [Sc, (6.16)] and its proof, [CK, pp. 13–14]).

In the above notation, for all  $\lambda, a$  and  $b$ , put

$$(2.13) \quad \begin{aligned} n_{\lambda} &:= \dim_{\mathbf{R}} H(Y; \lambda) = \dim_{\mathbf{C}} H_{\mathbf{C}}(Z; \lambda), \\ p_{\lambda}^{a,b} &:= \dim_{\mathbf{C}} P_{\lambda-2k}^{(\lambda)a+k, b+k} = \dim_{\mathbf{C}} Q_{\lambda-2k}^{(\lambda)a+k, b+\lambda-k}. \end{aligned}$$

Notice that, by construction, the middle terms and the terms on the extreme right hand side of the second equality in (2.13) are independent of  $k$  (cf. Remark (2.12)).

LEMMA (2.14). *For  $(\rho, r)$  as above, the following hold:*

- (i)  $\sum_{a+b=w-\lambda} p_{\lambda}^{a,b} = n_{\lambda} - n_{\lambda+2}$  for all  $0 \leq \lambda \leq w$ .
- (ii)  $p_{\lambda}^{b,a} = p_{\lambda}^{a,b}$  for all  $\lambda, a, b$  with  $0 \leq \lambda \leq w, a \geq 0, b \geq 0$  and  $a+b=w-\lambda$ .
- (iii)  $h^{a,b} = h^{a+1, b-1} - (p_0^{a+1, b-1} + p_1^{a+1, b-2} + \cdots + p_{b-1}^{a+1, 0}) + (p_0^{a,b} + p_1^{a-1, b} + \cdots +$

$p_a^{0,b}$ ) for all  $a, b$  with  $a \geq 0, b \geq 0$  and  $a + b = w$ .

PROOF. We first observe that there is an exact sequence

$$0 \longrightarrow P_\lambda^{(\lambda)} \longrightarrow H(Y; \lambda) \xrightarrow{N_+} H(Y; \lambda + 2) \longrightarrow 0$$

for every  $\lambda \geq 0$  (and  $N_-$  yields a right splitting). (i) and (ii) follow from this and (2.11).

In order to show (iii), we look at the morphism  $X_+ : H^{a+1, b-1} \rightarrow H^{a, b}$  and its kernel and cokernel:

$$\begin{aligned} \text{Ker} &= Q_0^{(0)a+1, b-1} \oplus Q_1^{(1)a+1, b-1} \oplus \dots \oplus Q_{b-1}^{(b-1)a+1, b-1} \\ &\stackrel{\subset}{=} P_0^{(0)a+1, b-1} \oplus P_1^{(1)a+1, b-2} \oplus \dots \oplus P_{b-1}^{(b-1)a+1, 0}, \\ \text{Coker} &\simeq Q_0^{(0)a, b} \oplus Q_{-1}^{(1)a, b} \oplus \dots \oplus Q_{-a}^{(a)a, b} \\ &\simeq Q_0^{(0)a, b} \oplus Q_1^{(1)a-1, b+1} \oplus \dots \oplus Q_a^{(a)0, b+a} \\ &\stackrel{\subset}{=} P_0^{(0)a, b} \oplus P_1^{(1)a-1, b} \oplus \dots \oplus P_a^{(a)0, b}. \end{aligned}$$

Looking at the dimension, we get (iii). ■

DEFINITION (2.15). We call a set of integers  $\{p_\lambda^{a, b}\}$ , which satisfies the conditions (i), (ii) and (iii) of (2.14), a set of primitive Hodge numbers belonging to  $\{h^{p, q}, n_\lambda\}$ .

**3. Admissible  $R$ -semi-simple elements.** We continue to use the notation in the previous sections.

PROPOSITION (3.1). Given a pair  $(Y, r) \in \mathfrak{g} \times D$ , there exists at most one representation  $\rho : SL_2(\mathbf{R}) \rightarrow G$  which is horizontal at  $r$  and  $\rho_*(y) = Y$ .

PROOF. Since  $y$  and  $z$  generate  $\mathfrak{sl}_2(\mathbf{C})$ , it is enough to show that if such a representation  $\rho$  exists then the eigenspaces of  $Z$ , and hence  $Z$  itself, are determined by the pair  $(Y, r)$ . Actually, we shall show by induction on the size  $w$  of the nest of diamonds of the Hodge- $(Z, X_\pm)$  decomposition (2.7) (cf. Remark (2.12)) that this nest of diamonds is completely determined by  $(Y, r)$ .

First notice that

$$(3.2) \quad Y = i(X_+ - X_-).$$

For a subspace  $M$  of  $H_{\mathbf{C}}$ , we put, throughout this proof,

$$\begin{aligned} M^\perp &:= \{v \in H_{\mathbf{C}} \mid S(v, \sigma u) = 0 \text{ for all } u \in M\}, \\ \text{projection } \{M \rightarrow H_r^{p, q}\} &:= \text{Im} \left\{ M \subset H_{\mathbf{C}} = \bigoplus_{p'+q'=w} H_r^{p', q'} \rightarrow H_r^{p, q} \right\}. \end{aligned}$$

Then we see that

$$\begin{aligned}
Q_w^{(w)0,w} &= \text{projection} \{ Y^w H_r^{w,0} \rightarrow H_r^{0,w} \}, \\
Q_{w-2k}^{(w)k,w-k} &= \text{projection} \{ Y^k Q_w^{(w)0,w} \rightarrow H_r^{k,w-k} \} \quad (0 \leq k \leq w), \\
\bigoplus_{0 \leq \lambda \leq w-1} Q_{-\lambda}^{(\lambda)w,0} &= H_r^{w,0} \cap (Q_{-w}^{(w)w,0})^\perp, \\
Q_{w-1}^{(w-1)1,w-1} &= \text{projection} \left\{ Y^{w-1} \left( \bigoplus_{0 \leq \lambda \leq w-1} Q_{-\lambda}^{(\lambda)w,0} \right) \rightarrow H_r^{1,w-1} \right\}, \\
Q_{w-1-2k}^{(w-1)1+k,w-1-k} &= \text{projection} \{ Y^k Q_{w-1}^{(w-1)1,w-1} \rightarrow H_r^{1+k,w-1-k} \} \quad (0 \leq k \leq w-1), \\
\bigoplus_{0 \leq \lambda \leq w-2} Q_{-\lambda}^{(\lambda)w,0} &= H_r^{w,0} \cap \left( \bigoplus_{w-1 \leq \lambda \leq w} Q_{-\lambda}^{(\lambda)w,0} \right)^\perp, \\
Q_{w-2}^{(w-2)2,w-2} &= \text{projection} \left\{ Y^{w-2} \left( \bigoplus_{0 \leq \lambda \leq w-2} Q_{-\lambda}^{(\lambda)w,0} \right) \rightarrow H_r^{2,w-2} \right\}, \\
Q_{w-2-2k}^{(w-2)2+k,w-2-k} &= \text{projection} \{ Y^k Q_{w-2}^{(w-2)2,w-2} \rightarrow H_r^{2+k,w-2-k} \} \quad (0 \leq k \leq w-2), \\
&\dots\dots\dots
\end{aligned}$$

Thus  $Q_{\lambda-2k}^{(\lambda)w-\lambda+k,\lambda-k}$  ( $0 \leq \lambda \leq w, 0 \leq k \leq \lambda$ ) are determined. Taking the complex conjugation by  $\sigma$  of these, we get  $Q_{-\lambda+2k}^{(\lambda)\lambda-k,w-\lambda+k} = \sigma Q_{\lambda-2k}^{(\lambda)w-\lambda+k,\lambda-k}$  ( $0 \leq \lambda \leq w, 0 \leq k \leq \lambda$ ). Applying the induction hypothesis to the nest of diamonds of size  $\leq w-2$  in

$$\left( \bigoplus_{\substack{0 \leq \lambda \leq w \\ 0 \leq k \leq \lambda}} (Q_{\lambda-2k}^{(\lambda)w-\lambda+k,\lambda-k} \oplus Q_{-\lambda+2k}^{(\lambda)\lambda-k,w-\lambda+k}) \right)^\perp$$

(cf. Remark (2.12)), we get our assertion. ■

**DEFINITION (3.3).** A pair  $(Y, r) \in \mathfrak{g} \times D$  is admissible if there exists a representation  $\rho: SL_2(\mathbf{R}) \rightarrow G$  which is horizontal at  $r$  and  $\rho_*(y) = Y$ .

The set of primitive Hodge numbers  $\{p_\lambda^{a,b}\}$  belonging to  $\{h^{p,q}, n_\lambda\}$  is called the type of an admissible pair  $(Y, r)$ .

$Y \in \mathfrak{g}$  is said to be admissible if  $(Y, r)$  is an admissible pair for some  $r \in D$ .

Now we prove the following numerical criterion for admissibility:

**THEOREM (3.4).**  $Y \in \mathfrak{g}$  is admissible if and only if  $Y$  is semi-simple over  $\mathbf{R}$  whose eigenvalues are contained in  $\{0, \pm 1, \pm 2, \dots, \pm w\}$  and there exists a set of primitive Hodge numbers  $\{p_\lambda^{a,b}\}$  belonging to  $\{h^{p,q}, n_\lambda\}$ , where  $n_\lambda := \dim H(Y; \lambda)$  (cf. Definition (2.15)).

**PROOF.** Since  $Y$  is semi-simple over  $\mathbf{R}$ , the eigenspaces  $H(Y; \lambda)$  are defined over  $\mathbf{R}$  and  $H(Y; \lambda)$  and  $H(Y; \mu)$  are  $S$ -orthogonal unless  $\lambda + \mu = 0$ . Therefore  $H(Y; \lambda)$  and  $H(Y; -\lambda)$  are  $S$ -dual.

Since  $n_{\lambda'} - n_{\lambda'+2} \geq 0$  for  $\lambda' \geq 0$  by the condition (2.14.i), we can take a direct sum

decomposition

$$(3.5) \quad H(Y; \lambda) = P_{\lambda}^{(\lambda)} \oplus P_{\lambda}^{(\lambda+2)} \oplus P_{\lambda}^{(\lambda+4)} \oplus \cdots \quad \text{for } \lambda \geq 0,$$

with  $\dim P_{\lambda}^{(\lambda+2k)} = n_{\lambda+2k} - n_{\lambda+2k+2}$ . Moreover, in the case  $\lambda = 0$ , the decomposition (3.5) can be taken to be  $S$ -orthogonal. We denote the  $S$ -dual decomposition by

$$(3.6) \quad H(Y; -\lambda) = P_{-\lambda}^{(\lambda)} \oplus P_{-\lambda}^{(\lambda+2)} \oplus P_{-\lambda}^{(\lambda+4)} \oplus \cdots \quad (\lambda \geq 0),$$

i.e.,  $P_{-\lambda}^{(\lambda+2k)}$  and  $P_{\lambda}^{(\lambda+2m)}$  are  $S$ -orthogonal unless  $k = m$ .

By the conditions (i) and (ii) of (2.14), we can choose a Hodge decomposition

$$(3.7) \quad C \otimes P_{\lambda}^{(\lambda+2k)} = \bigoplus_{\substack{a+b+2k=w-\lambda \\ a, b \geq 0}} P_{\lambda}^{(\lambda+2k)a+k, b+k} \quad \text{for } \lambda \geq 0, \quad k \geq 0,$$

with  $\dim P_{\lambda}^{(\lambda+2k)a+k, b+k} = p_{\lambda+2k}^{a, b}$ . Moreover, in the case  $\lambda = k = 0$ , the Hodge structure (3.7) can be chosen to be  $S$ -polarized. We denote the  $S(\cdot, \sigma \cdot)$ -orthogonal decomposition by

$$(3.8) \quad C \otimes P_{-\lambda}^{(-\lambda+2(\lambda+k))} = \bigoplus_{\substack{a+b+2\lambda+2k=w+\lambda \\ a, b \geq 0}} P_{-\lambda}^{(-\lambda+2(\lambda+k))a+\lambda+k, b+\lambda+k} \quad (\lambda \geq 0, k \geq 0),$$

i.e.,  $S(P_{-\lambda}^{(-\lambda+2(\lambda+k))a+\lambda+k, b+\lambda+k}, \sigma P_{\lambda}^{(\lambda+2k)a'+k, b'+k}) = 0$  unless  $(a, b) = (a', b')$ . Notice that  $P_{-\lambda}^{(-\lambda+2(\lambda+k))a+\lambda+k, b+\lambda+k} = P_{-\lambda}^{(\lambda+2k)a+\lambda+k, b+\lambda+k}$ .

Now we consider the cases  $\lambda \geq 0$  and  $\lambda < 0$  altogether. For  $k \geq \max\{-\lambda, 0\}$  and  $a \geq b$ , let

$$(3.9) \quad \{v_{\lambda, j}^{(\lambda+2k)a+k, b+k} \mid 1 \leq j \leq p_{\lambda+2k}^{a, b}\}$$

be a  $C$ -basis of  $P_{\lambda}^{(\lambda+2k)a+k, b+k}$  such that

$$(3.10) \quad S(v_{-\lambda, j}^{(-\lambda+2(\lambda+k))a+\lambda+k, b+\lambda+k}, \sigma v_{\lambda, j}^{(\lambda+2k)a+k, b+k}) = \delta_{jj'} (-1)^{a_i w - \lambda} \binom{\lambda+2k}{k}.$$

In the case  $a = b$ , we can moreover take the above basis (3.9) to consist of real elements. Put

$$(3.11) \quad v_{\lambda, j}^{(\lambda+2k)b+k, a+k} = \sigma v_{\lambda, j}^{(\lambda+2k)a+k, b+k} \quad (a \geq b).$$

Define now  $C$ -linear endomorphisms  $N_{\pm}$  of  $H_C$  by

$$(3.12) \quad \begin{aligned} N_+ v_{\lambda, j}^{(\lambda+2k)a+k, b+k} &:= k v_{\lambda+2, j}^{((\lambda+2)+2(k-1))a+k-1, b+k-1}, \\ N_- v_{\lambda, j}^{(\lambda+2k)a+k, b+k} &:= (\lambda+k) v_{\lambda-2, j}^{((\lambda-2)+2(k+1))a+k+1, b+k+1}, \end{aligned}$$

for all  $\lambda$ , non-negative  $a, b$  and  $k \geq \max\{-\lambda, 0\}$ . By construction, it is easy to see that  $N_{\pm}$  commute with the complex conjugation  $\sigma$  and satisfy the commutation relations:  $[N_+, N_-] = Y$ , and  $[Y, N_{\pm}] = \pm 2N_{\pm}$ , respectively. It is also easy to verify that  $S(N_{\pm} \cdot, \cdot) + S(\cdot, N_{\pm} \cdot) = 0$ , respectively. Indeed, for example, one can compute as

$$\begin{aligned}
& S(N_+ v_{-\lambda, j}^{(-\lambda+2(\lambda+k))a+\lambda+k, b+\lambda+k}, \sigma v_{\lambda-2, j'}^{((\lambda-2)+2(k+1))a+k+1, b+k+1}) \\
& + S(v_{-\lambda, j}^{(-\lambda+2(\lambda+k))a+\lambda+k, b+\lambda+k}, N_+ \sigma v_{\lambda-2, j'}^{((\lambda-2)+2(k+1))a+k+1, b+k+1}) \\
& = \delta_{jj'} (-1)^a i^{w-\lambda+2} \frac{(\lambda+k)(\lambda+k-1)!(k+1)!}{(\lambda+2k)!} + \delta_{jj'} (-1)^a i^{w-\lambda} \frac{(k+1)k!(\lambda+k)!}{(\lambda+2k)!} = 0.
\end{aligned}$$

Thus we see that  $N_{\pm} \in \mathfrak{g}$  and hence there exists a unique representation

$$(3.13) \quad \rho: SL_2(\mathbf{R}) \rightarrow G \quad \text{such that} \quad \rho_*(Y) = Y \quad \text{and} \quad \rho_*(n_{\pm}) = N_{\pm}, \quad \text{respectively.}$$

By using the Cayley element  $c := \rho(c_1) \in G_c$ , we define

$$(3.14) \quad Q_{\lambda}^{(\lambda+2k)a+k, b+\lambda+k} := c P_{\lambda}^{(\lambda+2k)a+k, b+k}, \quad H^{p,q} := \bigoplus_{\substack{a+k=p \\ b+\lambda+k=q}} Q_{\lambda}^{(\lambda+2k)a+k, b+\lambda+k},$$

where, on the right hand side of the second equality, the summation is taken over all the eigenvalues  $\lambda$  of  $Y$ , all integers  $k \geq \max\{-\lambda, 0\}$  and all non-negative integers  $a, b$  with  $a+b+\lambda+2k=w$ . This defines a Hodge structure. Indeed, by using (1.20.ii), one sees that

$$\begin{aligned}
\sigma Q_{\lambda}^{(\lambda+2k)a+k, b+\lambda+k} &= \sigma c P_{\lambda}^{(\lambda+2k)a+k, b+k} = c^{-1} \sigma P_{\lambda}^{(\lambda+2k)a+k, b+k} \\
&= c^{-1} P_{\lambda}^{(\lambda+2k)b+k, a+k} = c P_{-\lambda}^{(-\lambda+2(\lambda+k))b+\lambda+k, a+\lambda+k} = Q_{-\lambda}^{(-\lambda+2(\lambda+k))b+\lambda+k, a+k},
\end{aligned}$$

and hence  $\sigma H^{p,q} = H^{q,p}$ . One can moreover verify that (3.14) is  $S$ -polarized. Indeed, the direct sum in (3.14) is  $S$ -orthogonal by construction and, for

$$cv_{\lambda, j}^{(\lambda+2k)a+k, b+k}, \quad cv_{\lambda, j'}^{(\lambda+2k)a+k, b+k} \in Q_{\lambda}^{(\lambda+2k)a+k, b+\lambda+k} \subset H^{p,q},$$

one can compute as

$$\begin{aligned}
& i^{p-q} S(cv_{\lambda, j}^{(\lambda+2k)a+k, b+k}, \sigma cv_{\lambda, j'}^{(\lambda+2k)a+k, b+k}) \\
& = i^{a-b-\lambda} S(cv_{\lambda, j}^{(\lambda+2k)a+k, b+k}, c^{-1} \sigma v_{\lambda, j'}^{(\lambda+2k)a+k, b+k}) \\
& = i^{a-b-\lambda} S(c^2 v_{\lambda, j}^{(\lambda+2k)a+k, b+k}, \sigma v_{\lambda, j'}^{(\lambda+2k)a+k, b+k}) \\
& = i^{a-b-\lambda+2k} S(v_{-\lambda, j}^{(-\lambda+2(\lambda+k))a+\lambda+k, b+\lambda+k}, \sigma v_{\lambda, j'}^{(\lambda+2k)a+k, b+k}) \\
& = \delta_{jj'} i^{a-b+2k+2a+w-\lambda} \left/ \binom{\lambda+2k}{k} \right/ = \delta_{jj'} \left/ \binom{\lambda+2k}{k} \right/.
\end{aligned}$$

Thus we have  $\{H^{p,q}\} \in D$ .

Finally, we claim that the representation  $\rho$  in (3.13) is horizontal at  $\{H^{p,q}\} \in D$ . Indeed, since  $Z = (\text{Ad } c)Y$ ,  $X_{\pm} = (\text{Ad } c)N_{\pm}$ , one can compute, by (1.20), as

$$\begin{aligned}
Z Q_{\lambda}^{(\lambda+2k)a+k, b+\lambda+k} &= c Y P_{\lambda}^{(\lambda+2k)a+k, b+k} = Q_{\lambda}^{(\lambda+2k)a+k, b+\lambda+k}, \\
X_{\pm} Q_{\lambda}^{(\lambda+2k)a+k, b+\lambda+k} &= c N_{\pm} P_{\lambda}^{(\lambda+2k)a+k, b+k}
\end{aligned}$$

$$= cP_{\lambda \pm 2}^{((\lambda \pm 2) + 2(k \mp 1))a + k \mp 1, b + k \mp 1} = Q_{\lambda \pm 2}^{((\lambda \pm 2) + 2(k \mp 1))a + k \mp 1, b + \lambda + k \pm 1}.$$

This completes the proof of the theorem.  $\blacksquare$

We remark that the condition on  $\{n_\lambda\}$  in Theorem (3.4) coincides with the one in [CK, (2.20)] in the case of weight 2.

Fix identifications  $D \simeq G/V$  and  $R \simeq G/K$ , where  $K$  is a maximal compact subgroup of  $G$  containing  $V$  and  $R$  is the associated Riemannian symmetric domain, and let  $\theta_K$  be the associated Cartan involution. We denote the projection by

$$(3.15) \quad \pi: D \simeq G/V \rightarrow G/K \simeq R.$$

PROPOSITION (3.16). *We use the notation in Theorem (3.4). Let  $Y \in \mathfrak{g}$  be an admissible element.*

(i) *If there exists  $r \in \pi^{-1}([K])$  such that  $(Y, r)$  is an admissible pair, then  $\theta_r Y = -Y$ , where  $\theta_r$  is the Cartan involution on  $\mathfrak{g}$  induced from (1.11).*

(ii) *If  $\theta_K Y = -Y$ , then there exists  $r \in \pi^{-1}([K])$  such that  $(Y, r)$  is an admissible pair.*

(iii) *For each set of primitive Hodge numbers  $\{p_\lambda^{a,b}\}$  belonging to  $\{h^{p,q}, n_\lambda\}$ ,  $G_Y := \{g \in G \mid (\text{Ad } g)Y = Y\}$  acts transitively on the set  $\{r \in D \mid (Y, r) \text{ is an admissible pair of type } \{p_\lambda^{a,b}\}\}$ .*

PROOF. (i) follows from (3.2) and (1.11).

(ii): Assume  $\theta_K Y = -Y$ . Take a point  $r' \in D$  at which  $Y$  is admissible and let  $K'$  be the maximal compact subgroup of  $G$  associated to the Cartan involution  $\theta_{r'}$ . By the result in (i) for  $(Y, r')$  and the assumption,  $Y$  can be viewed as a tangent vector to  $R$  at  $[K']$  as well as at  $[K]$ :  $Y \in T_R([K'])$ ,  $Y \in T_R([K])$ . By the transitivity of tangent spaces of a Riemannian symmetric domain, there exists  $g \in G$  such that  $(\text{Int } g)K' = K$  and  $(\text{Ad } g)Y = Y$ . Hence the admissibility of  $(Y, r')$  implies that of  $((\text{Ad } g)Y, gr') = (Y, gr')$ , where  $gr' \in \pi^{-1}([K])$ .

(iii): Suppose that  $r, r' \in D$  are points at which  $Y$  is admissible of the same type  $\{p_\lambda^{a,b}\}$ . Let  $\rho, \rho': SL_2(\mathbf{R}) \rightarrow G$  be the corresponding representations. It is enough to show that there exists  $g \in G$  such that  $\rho' = (\text{Int } g)\rho$ . Indeed, if this is the case, then  $(\text{Ad } g)Y = (\text{Ad } g)(\rho_*(y)) = \rho'_*(y) = Y$  and  $gr = g\tilde{\rho}(i) = \tilde{\rho}'(i) = r'$ .

We can construct such a  $g \in G$  elementarily by using bases of  $H_c$  according to the  $S$ -polarized Hodge- $(Z, X_\pm)$  decompositions, where  $(Z, X_\pm) = (\rho_*(z), \rho_*(x_\pm))$ ,  $(\rho'_*(z), \rho'_*(x_\pm))$ . Thus we get our assertion.  $\blacksquare$

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